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Satake superdiagrams, real forms and Iwasawa decomposition of classical Lie superalgebras

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Abstract. Satake superdiagrams corresponding to the Lie superalgebras $A(m,n)$, $B(m,n)$, $B(o,n)$, $C(n)$, $D(m,n)$ together with the exceptional ones $D(2,1,\alpha)$, $F(4)$, $G(3)$ are constructed from their Dynkin diagrams with a view to determine the real forms of these superalgebras. The involutive automorphisms are computed from a modified formula to account for all pertinent cases in a consistent way. This mechanism is also used to furnish a general treatment of the Iwasawa and Langlands decompositions of these superalgebras. In particular, we compute these decompositions for $D(2,1,\alpha)$ for illustrational purposes and obtain the corresponding induced representations using the method of Schmidt construction.

1. Introduction

The main motivation to undertake this investigation stems from the incredible power and mathematical sophistication of the techniques of supersymmetry in the realms of particle physics and field theory whereby the bosons and fermions are treated in a unified manner. Since the study of Lie groups and their associated Lie algebras constitute a concrete realization of the tremendous proliferation of the various symmetry transformations, it is only natural to visualize, evaluate and interpret the possible consequences of a suitable supersymmetric extension of these algebras. The constructs so generated are the Lie superalgebras [1–3] or graded Lie algebraic structures which can be readily transcribed for the purpose of application to both bosonic and fermionic sectors in a systematic and consistent framework. It is worth recalling that the essential elements entering one of the straightforward methods for the study of ordinary Lie algebras comprise the root system (up to a transformation of the Weyl group) and the Dynkin diagrams which in turn can be used to construct the corresponding Satake diagrams [4–7] from which to evaluate the involutive root automorphisms. The evaluation of root automorphisms for the Lie superalgebras is achieved in exactly the same manner as for the ordinary Lie algebras, by constructing the Satake superdiagrams from the corresponding Dynkin diagrams for these superalgebras.

In this paper we address the basic or contragradient superalgebras [1, 2], namely, the $A(m,n)$, $B(m,n)$, $B(o,n)$, $C(n)$, $D(m,n)$ and the exceptional Lie superalgebras $F(4)$, $G(3)$ as well as $D(2,1,\alpha)$ which evidently are closely linked to the usual simple Lie algebras. The bosonic and fermionic sectors are, however, characterized by the even and odd roots, respectively. In contrast to the case of simple Lie algebras where there is only one simple root system, the superalgebraic structures are, in general, endowed with several unequivalent simple root systems. The involutive automorphisms of these algebras are obtained from Satake superdiagrams with a modified formula and their real forms are determined explicitly

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[8–10]. Although the forms are real, their representations can, in general, be real, antireal and areal. We also carry out the Iwasawa [11] and Langlands [12] decompositions of the exceptional Lie superalgebra $D(2,1,\alpha)$ within this scheme.

This paper is organized as follows. In section 2, we give a brief summary of the classification of classical Lie superalgebras along with their root systems and Dynkin diagrams, displayed in table 1. Section 3 contains an outline of the essential steps for the construction of Satake superdiagrams from the Dynkin diagrams of the Lie superalgebras, which are listed in table 2. Section 4 is devoted to a general formulation of the Iwasawa and Langlands decompositions of these superalgebras following the procedure analogous to that of ordinary Lie algebras. As an illustration, in section 5 we consider the Satake superdiagrams of $D(2,1,\alpha)$ corresponding to its three distinct real forms and explicitly construct the Iwasawa and Langlands decompositions of $D(2,1,\alpha)$. It is shown that this gives rise to parabolic subalgebras which serve as a basis for obtaining the corresponding induced representation through Schmidt [13] construction. Finally, we discuss the results and conclusions in section 6.

2. Root systems and Dynkin diagrams of Lie superalgebras

The Lie bracket in a Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is defined by the equality

$$[a,b] = ab - (-1)^{|a||b|}ba \qquad \forall a,b \in \mathcal{G} \tag{2.1}$$

where $|a|$ is the degree of a, being 0 for the elements of the subalgebra $\mathcal{G}_0$ and 1 for the elements of $\mathcal{G}_1$. Moreover, we have globally

$$[\mathcal{G}_0,\mathcal{G}_0] \subset \mathcal{G}_0 \qquad [\mathcal{G}_0,\mathcal{G}_1] \subset \mathcal{G}_1 \qquad [\mathcal{G}_1,\mathcal{G}_1] \subset \mathcal{G}_0. \tag{2.2}$$

Let H be a Cartan subalgebra of $\mathcal{G}_0$. A root α of $\mathcal{G}(\alpha \neq 0)$ will be an element $\alpha \in H^*$, the dual of H, such that

$$\mathcal{G}_\alpha = \{e_\alpha \in \mathcal{G} | [e_\alpha, h] = \alpha(h)e_\alpha, h \in H\}. \tag{2.3}$$

A root α is called even if $\mathcal{G}_0 \cap \mathcal{G}_\alpha \neq 0$ and odd if $\mathcal{G}_1 \cap \mathcal{G}_\alpha \neq 0$. Different families of basic classical Lie superalgebras can now be introduced along with their root systems; Δ_0 and Δ_1 being the set of even and odd roots respectively. These are:

2.1. Classical Lie superalgebras

(1) $A(m,n) = sl(m+1,n+1)$.

This is a type 1 Lie superalgebra defined by

$$A(m,n) = A_{-1}(m,n) + A_0(m,n) + A_1(m,n) \tag{2.4}$$

in which the ordinary Lie algebra $A_0(m,n)$ is reducible and is given by

$$A_0(m,n) = A_m \oplus A_n \oplus K \tag{2.5}$$

where K is a real number corresponding to the Abelian group $U(1)$. The root system of $A(m,n)$ is given in terms of $\epsilon_1,\ldots,\epsilon_{m+1},\delta_1,\ldots,\delta_{n+1}$ as

$$\begin{aligned}\Delta_0 &= \{\epsilon_i - \epsilon_j, \delta_i - \delta_j\}\\ \Delta_1 &= \{\pm(\epsilon_i - \delta_i)\}.\end{aligned} \tag{2.6}$$

(2) $B(m,n) = osp(2m+1,2n)$.

This is a type 2 Lie superalgebra defined by

$$B(m,n) = B_0(m,n) + B_1(m,n) \tag{2.7}$$

where the ordinary Lie algebra $B_0(m,n)$ is given by

$$B_0(m,n) = B_m \oplus C_n. \tag{2.8}$$

The root system is given by

$$\begin{aligned} \Delta_0 &= \{\pm\epsilon_i \pm \epsilon_j, \pm 2\delta_i, \pm\epsilon_i, \pm\delta_i \pm \delta_j\} \qquad i \neq j \\ \Delta_1 &= \{\pm\delta_i, \pm\epsilon_i \pm \delta_j\}. \end{aligned} \tag{2.9}$$

(3) $B(o,n) = osp(1,2n)$.
This is also a type 2 Lie superalgebra defined by

$$B(o,n) = B_0(o,n) \oplus B_1(o,n) \tag{2.10}$$

where

$$B_0(o,n) = C_n. \tag{2.11}$$

The root system is given by

$$\begin{aligned} \Delta_0 &= \{\pm\delta_i \pm \delta_j, \pm 2\delta_i\} \qquad i \neq j \\ \Delta_1 &= \{\pm\delta_i\}. \end{aligned} \tag{2.12}$$

(4) $C(n) = osp(2n,2)$.
This again is a type 2 Lie superalgebra defined by

$$C(n) = C_{-1}(n) + C_0(n) + C_1(n) \tag{2.13}$$

where

$$C_0(n) = C_n + D_1. \tag{2.14}$$

The root system is given in terms of $\epsilon_1, \delta_1, \ldots, \delta_{n-1}$ as

$$\begin{aligned} \Delta_0 &= \{\pm 2\delta_i, \pm\delta_i \pm \delta_j\} \\ \Delta_i &= \{\pm\epsilon_1 \pm \delta_i\}. \end{aligned} \tag{2.15}$$

(5) $D(m,n) = osp(2m,2n)$.
This is a type 2 Lie superalgebra defined by

$$D(m,n) = D_0(m,n) + D_1(m,n) \tag{2.16}$$

where

$$D_0(m,n) = D_m + C_n. \tag{2.17}$$

The root system is given by

$$\begin{aligned} \Delta_0 &= \{\pm\epsilon_i \pm \epsilon_j, \pm 2\delta_i, \pm\delta_i \pm \delta_j\} \qquad i \neq j \\ \Delta_1 &= \{\pm\epsilon_i \pm \delta_j\}. \end{aligned} \tag{2.18}$$

2.2. Exceptional Lie superalgebras

(6) $D(2,1,\alpha) : \alpha \in k^*/\{0,-1\}$.
This one-parameter family of 17-dimensional Lie superalgebra is given by

$$D(2,1,\alpha) = D_0(2,1,\alpha) + D_1(2,1,\alpha) \tag{2.19}$$

where

$$D_0(2,1,\alpha) = A_1 + A_1 + A_1. \tag{2.20}$$

The roots are expressed in terms of linear functions $\epsilon_1, \epsilon_2, \epsilon_3$ as

$$\begin{aligned} \Delta_0 &= \{\pm 2\epsilon_i\} \\ \Delta_1 &= \{\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3\}. \end{aligned} \tag{2.21}$$

(7) $F(4)$.

This is a 40-dimensional Lie superalgebra defined by

$$F(4) = F_0(4) + F_1(4) \tag{2.22}$$

where

$$F_0(4) = A_1 + B_3. \tag{2.23}$$

The roots are expressed in terms of $\epsilon_1, \epsilon_2, \epsilon_3$ corresponding to B_3 and δ corresponding to A_1 as

$$\begin{aligned} \Delta_0 &= \{\pm\delta, \pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i\} \\ \Delta_1 &= \{\tfrac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_2 \pm \delta)\}. \end{aligned} \tag{2.24}$$

(8) $G(3)$.

This is a 31-dimensional Lie superalgebra defined by

$$G(3) = G_0(3) + G_1(3) \tag{2.25}$$

where

$$G_0(3) = A_1 + G_2. \tag{2.26}$$

The roots are expressed in terms of δ and $\epsilon_1, \epsilon_2, \epsilon_3$ with the condition $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$, as

$$\begin{aligned} \Delta_0 &= \{\pm 2\delta, \epsilon_i - \epsilon_j, \pm\epsilon_i\} \\ \Delta_1 &= \{\pm\delta, \pm\epsilon_i \pm \delta\}. \end{aligned} \tag{2.27}$$

As for the ordinary Lie algebras, here also we can define a system of simple roots $\pi = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Delta$ if every other root of Lie superalgebra can be obtained as a linear combination of these systems of simple roots. Now it is always possible to define a $r \times r$ Cartan matrix $A = (a_{ij})$ associated with a set of simple roots with the following conditions:

$$\begin{aligned} &[e_{\pm\alpha_i}, h_j] = \pm a_{ij} e_{\pm\alpha_j} \\ &[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} h_j \\ &[h_i, h_j] = 0 \end{aligned} \tag{2.28}$$

the $h_1, h_2, \ldots, h_r$ generating the corresponding Cartan subalgebra H. The a_{ij} are the elements of the Cartan matrix which can be chosen to be symmetric and defined as follows

$$a_{ij} = (\alpha_i, \alpha_j). \tag{2.29}$$

Such a Cartan matrix can be obtained [2] from a non-symmetric Cartan matrix by multiplying a diagonal matrix corresponding to the particular superalgebra.

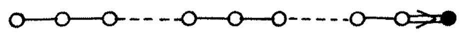
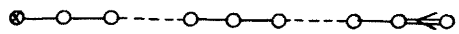
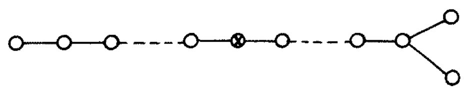
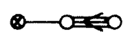
We now associate to each simple root system of $\mathcal{G}$, a Dynkin diagram according to the following rules.

(i) To each simple *bosnic root* α_i we associate an open circle, to each simple *fermionic root* α_i we associate a grey circle, $\otimes$, if $a_{ii} = 0$ and a full circle $\bullet$ if $a_{ii} \neq 0$.

(ii) The ith and jth circles will be joined by η_{ij} lines with

$$\eta_{ij} = \begin{cases} \dfrac{2|a_{ij}|}{\min(|a_{ii}|, |a_{jj}|)} & \text{if } a_{ii} \cdot a_{jj} \neq 0 \\ 2\dfrac{|a_{ij}|}{\min a_{kk} \neq 0 |a_{kk}|} & \text{if } a_{ii} \neq 0,\ a_{jj} = 0 \\ |a_{ij}| & \text{if } a_{ii} = a_{jj} = 0. \end{cases} \tag{2.30}$$

Table 1. Dynkin diagrams corresponding to the basic classical Lie superalgebras.

Classical Lie superalgebras	Dynkin diagrams
A(m, n)	
B(m, n)	
B(0, n)	
C(n)	
D(m, n)	
D(2, 1, α)	
F(4)	
G(3)	

(iii) We add an arrow on the lines connecting the ith and jth circles when $|\eta_{ij}| > 1$, pointing from i to j if $a_{ii} \cdot a_{jj} \neq 0$ and $|a_{jj}|$ or if $a_{ii} = 0, a_{jj} \neq 0, |a_{jj}| < 2$ and pointing from j to i if $a_{ii} = 0, a_{jj} \neq 0, |a_{jj}| > 2$.

We note that for a given Lie superalgebra there may be associated a number of non-equivalent simple root systems. Therefore, not only one Dynkin diagram but a number of non-equivalent Dynkin diagrams are possible. In this paper, we choose the Dynkin diagrams related to a special simple root system called the distinguished root system which contains the smallest number of fermionic roots. These are illustrated in table 1.

3. Construction of Satake superdiagrams

As an analogue of symmetric space, we consider the homogeneous supermanifold $\mathcal{G}/\mathcal{G}^\sigma$, where $\mathcal{G}$ is a real Lie superalgebra and σ is an involutive automorphism of $\mathcal{G}$. This is called symmetric superspace [9, 14]. In this case, however, we have a weak analogue of the notion of compactness, because there are no real simple Lie superalgebras for which the restriction of the invariant metric to the even part has a definite sign. A symmetric superspace with compact base will be called compact.

Let M be a compact semisimple symmetric superspace, G its supergroup of motion, K the isotropy group of the base point, A a maximal torus in M and $T \supset Q(A)$ a maximal torus in G. Let R be the root system of G relative to T and R_- the root system of M relative to A. For $\alpha \in R$, let $\bar{\alpha} = \{(-1)^{|\alpha|}\alpha - \sigma(\alpha)\}$, where σ is the involutive automorphism of α. Then $R_- = \{\bar{\alpha} | \bar{\alpha} \neq 0, \alpha \in R\}$. Also let $R_0 = \{\alpha \epsilon R | \bar{\alpha} = 0\}$. Further, let B_- (resp. B) denote the basis of R_- (resp. R), then B_0 will be a basis of R_0, where $B_0 = B \cap R_0$.

Proposition. Let $B/B_0 = \{\alpha_1, \dots, \alpha_r\}$ and $B_0 = \{\beta_1, \dots, \beta_S\}$, then

$$-\sigma(\alpha_i) = \alpha_{\pi(i)} + (-1)^{|\alpha_i|} \sum \eta_{il}\beta_l \tag{3.1}$$

where π is an involutive permutation of $\{1, 2, \ldots, r\}$ and $(-1)^{|\alpha_i|}\eta_{il}$ are non-negative integers.

Proof. Let

$$-\sigma(\alpha_i) = (-1)^{|\alpha|} \sum C_{ij}\alpha_j. \tag{3.2}$$

As all types of automorphisms must respect grading, i.e. $|\alpha_i| = |\alpha|$, so we have

$$-\sigma(\alpha_i) = (-1)^{|\alpha_i|} \sum_{j=1}^{r} m_{ij}\alpha_j + (-1)^{|\alpha_i|} \sum_{l=1}^{S} \eta_{il}\beta_l. \tag{3.3}$$

Now applying $\alpha_i \to -\sigma(\alpha_i)$ to this identity, we obtain

$$\begin{aligned}(-\sigma)^2(\alpha_i) &= (-1)^{|\alpha_l|} \sum m_{ij}(-1)^{|\alpha_j|} \sum m_{jk}\alpha_k \\ &\quad + (-1)^{|\alpha_i|} \sum ((-1)^{|\alpha_j|} m_{ij}\eta_{jl} - (-1)^{|\beta_l|}\eta_{il})\beta_l.\end{aligned} \tag{3.4}$$

Since σ is involutive, the left-hand side of the above equation is equal to α_i and we have

$$\sum (-1)^{|\alpha_i|} m_{ij}(-1)^{|\alpha_j|} m_{jk} = \delta_{ik}. \tag{3.5}$$

So, the terms $(-1)^{|\alpha_j|}m_{ij}$ can be considered as the elements of the permutation matrix and we finally obtain the desired result,

$$-\sigma(\alpha_i) = \alpha_{\pi(i)} + (-1)^{|\alpha_i|} \sum \eta_{il}\beta_l.$$

Now $B_- = \{\bar{\alpha} | \alpha \in B/B_0\}$, and we should note that

$$\sigma(\beta_i) = (-1)^{|\beta_i|}\beta_i \qquad \text{with } \alpha + \sigma(a) \in R \qquad \forall \alpha \in R.$$

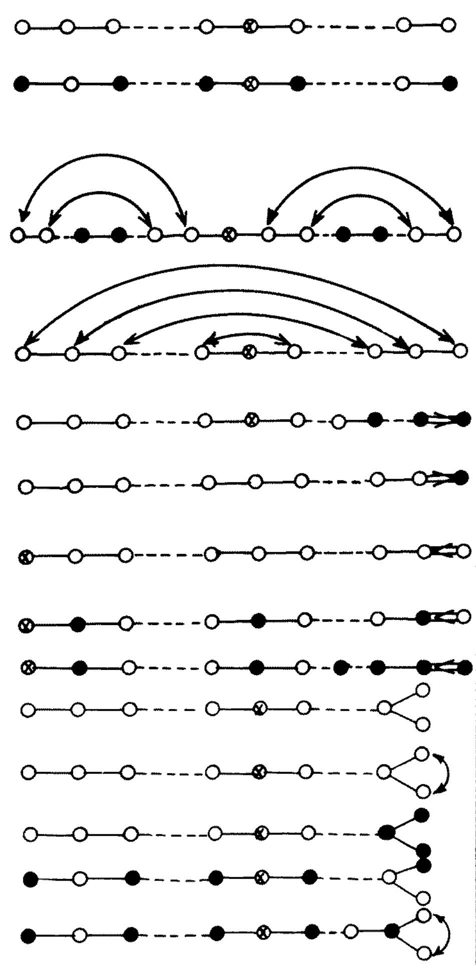
We can now associate with B its Satake superdiagrams. In the Dynkin diagram of B denote the roots α_i by usual open, grey and full circles and the roots β_l by full circles ●. We should note that this full circle is different from the full circle root associated with a non-degenerate odd root such as in $B(o, n)$, for instance. If $\pi(i) = k$, then it will be indicated by $\curvearrowleft\!\curvearrowright$.

Satake superdiagrams determine the involution σ of R uniquely. However, in some cases when the grey root is blackened, we cannot solve for η_{il} uniquely. We should therefore avoid these types of Satake superdiagrams. A simple Lie superalgebra over R is determined up to isomorphism by its Satake superdiagrams. In other words, Satake superdiagrams will correspond to all possible real forms of Lie superalgebras. If we restrict ourselves to those real Lie superalgebras for which the representation is real, then we obtain exactly the same real forms of Lie superalgebras as given by Parker [8]. It is worth mentioning a theorem due to him in this context. Up to isomorphism, the real forms of the classical Lie superalgebras are uniquely determined by the real form $\mathcal{G}_{oc}$ of the Lie subalgebra $\mathcal{G}_0$. These are listed in table 2. □

4. Iwasawa and Langlands decompositions of Lie superalgebras

We now extend the notion of direct determination of Iwasawa [11] and Langlands [12] decompositions of Lie algebras to the case of Lie superalgebras. Let $\tilde{\mathcal{G}}$ be a real Lie superalgebra generated by its compact real form $\tilde{\mathcal{G}}_k$ by an involutive automorphism defined

Table 2. Satake superdiagrams corresponding to the real forms of basic classical Lie superalgebras.

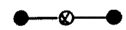
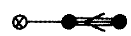
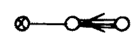
Classical Lie superalgebras	Even part of real form of Lie superalgebras	Satake superdiagrams
$A(m-1,n-1)$	(i) $sl(m,R)\oplus sl(n,R)\oplus R$	
	(ii) $su^*(m)\oplus su^*(n)\oplus R$ if m,n are even.	
	(iii) $su(p,m-p)\oplus su(r,n-r)\oplus iR$	
	(iv) $sl(n,c)$ if $m=n$	
$B(m,n)$	$sp(m,R)\oplus so(p,q)$	
$B(0,n)$	$sp(m,R)$	
$C(n)$	(i) $sp(m,R)\oplus so(2)$	
	(ii) $sp(r,s)\oplus so(2)$	
$D(m,n)$	(i) $sp(m,R)\oplus so(p,q)$	
	(ii) $sp(r,s)\oplus so^*(2p)$	

with respect to the Cartan subalgebra $\tilde{h}$ of $\tilde{\mathcal{G}}_c$, $\tilde{\mathcal{G}}_C$ being the complexification of $\tilde{\mathcal{G}}$. The following commutation relations are satisfied by the elements of $\tilde{\mathcal{G}}_c$

$$
\begin{aligned}
&[e_\alpha, h] = \alpha(h)e_\alpha \qquad h\in\tilde{h} \qquad \alpha\in\Delta\\
&[e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta}e_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root}\\ 0 & \text{otherwise}\end{cases}\\
&[e_\alpha, e_{-\alpha}] = h_\alpha \qquad h_\alpha\in\tilde{h}.
\end{aligned} \tag{4.1}
$$

Here Δ denotes the set of roots of $\tilde{\mathcal{G}}_C$ with respect to $\tilde{h}$ and the Killing form is defined as $B(e_\alpha, e_{-\alpha}) = -1$, $a(h) = B(h, h_\alpha)$. The compact real form $\tilde{\mathcal{G}}_k$ may be taken to consist of

Table 2. (Continued)

D(2, 1, α)	(i) sl(2, R)⊕sl(2, R)⊕ sl(2, R)	
	(ii) su(2)⊕sl(2, R)⊕ su(2)	
	(iii) sl(2, C)⊕sl(2, R)	
F(4)	(i) sl(2, R)⊕so(7)	
	(ii) sl(2, R)⊕so(3, 4)	
	(iii) su(2)⊕so(2, 5)	
	(iv) su(2)⊕ so(1, 6)	
G(3)	(i) sl(2, R)⊕ $G_{2,0}$	
	(ii) sl(2, R)⊕ $G_{2,2}$	

$\mathrm{i}h_\alpha, \alpha = \alpha_1, \alpha_2, \ldots, \alpha_r$, where r is the rank of $\mathcal{G}$ and $(e_\alpha + e_{-\alpha}), \mathrm{i}(e_\alpha - e_{-\alpha}) \forall\alpha$. Let $\tilde{k}$ be the maximal compact subalgebra of $\tilde{\mathcal{G}}$ defined such that $a \in \tilde{k}$ iff $a \in \tilde{\mathcal{G}}_C$ and $\sigma a = a$. Let $\tilde{p}$ be the subspace of $\tilde{\mathcal{G}}$ such that $a \in \tilde{p}$ iff $a \in \tilde{\mathcal{G}}_C$ and $\sigma a = -a$. Thus $\tilde{k}$ and $\tilde{p}$ are given by

$$\tilde{k} = \{\mathrm{i}h_\alpha, \alpha = \alpha_1, \alpha_2, \ldots, \alpha_r \text{ and } (e_\alpha + e_{-\alpha}), \mathrm{i}(e_\alpha - e_{-\alpha}) \forall\alpha \,|\, \exp\alpha(h) = 1\} \tag{4.2}$$

$$\tilde{p} = \{\mathrm{i}(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) \forall\alpha \,|\, \exp\alpha(h) = -1\}. \tag{4.3}$$

Let $\tilde{a}$ be the maximal Abelian subalgebra of $\tilde{p}$ with dimension $|m_1|$ and $\tilde{m}$ be the centralizer of $\tilde{a}$ in $\tilde{k}$. Complexification of $\tilde{a} \oplus \tilde{m}$ gives a Cartan subalgebra $\tilde{h}'$ of $\tilde{\mathcal{G}}_C$ with basis $\{H_1', H_2', \ldots, H_r'\}$. There exists an inner automorphism $V : \tilde{h}' \to \tilde{h}$, i.e.

$$H_j = V H_j' \qquad \text{where } V = \pi_\alpha \nu_\alpha \qquad a \in \Delta. \tag{4.4}$$

Let Δ^+ be the set of positive roots, then

$$h_\alpha = \sum_{j=1}^{r} b_j(\alpha) H_j. \tag{4.5}$$

Thus $\alpha \in \Delta^+$ iff $b_j(\alpha) > 0$ where j is the least index such that $b_j(\alpha) \neq 0$. The positive roots can again be divided into the following classes:

$$\begin{aligned} &\text{(i) } \Delta_+^+ = \{\alpha | \alpha \in \Delta^+, \alpha(h) \neq \alpha(V\sigma V^{-1}h) \ \forall h \in \tilde{h}\} \\ &\text{(ii) } \Delta_-^+ = \{\alpha | \alpha \in \Delta^+, \alpha(h) = \alpha(V\sigma V^{-1}h) \ \forall h \in \tilde{h}\}. \end{aligned} \tag{4.6}$$

Let the subalgebra $\tilde{\tilde{n}}$ be spanned by the elements $V^{-1}e_\alpha$ for $\alpha \in \Delta_+^+$ and $\tilde{n} = \tilde{\tilde{n}} \cap \tilde{\mathcal{G}}$, where $\tilde{\tilde{n}}$ and $\tilde{n}$ are the nilpotent subalgebras of $\tilde{\mathcal{G}}_C$ and $\tilde{\mathcal{G}}$, respectively. Thus, the Iwasawa decomposition of $\tilde{\mathcal{G}}$ is given by

$$\tilde{\mathcal{G}} = \tilde{k} \oplus \tilde{a} \oplus \tilde{n}. \tag{4.7}$$

A minimal parabolic subalgebra is defined to be any subalgebra that is conjugate to

$$\tilde{p}_1 = \tilde{m} \oplus \tilde{a} \oplus \tilde{n}. \tag{4.8}$$

Any subalgebra of $\tilde{\mathcal{G}}$ containing the minimal parabolic subalgebra is a general parabolic subalgebra. There exist $2^{|m_1|}$ classes of parabolic subalgebras of $\tilde{\mathcal{G}}$ and in each such class there is a standard parabolic subalgebra which can be obtained through the following prescription. Let Σ be the set of roots for $\tilde{a}$ and ψ be the set of positive roots in Σ. Let θ denote the subset of ψ and $\langle\theta\rangle$ the set of roots in Σ which arises as a linear combination of roots in θ. Define $\langle\theta\rangle_\pm = \Sigma_\pm \cap \langle\theta\rangle$, where Σ_+ and Σ_- denote the positive and negative roots in Σ, respectively. Let $\eta_+(\theta)$, $\eta_-(\theta)$ and $\eta(\theta)$ denote the subspaces of $\tilde{a}$ corresponding to $\langle\theta\rangle_+$, $\langle\theta\rangle_-$ and $\Sigma_\pm - \langle\theta\rangle_\pm$, respectively. We now define

$$a_\theta = \{a \in \tilde{a} | \lambda(a) = 0\ \forall\lambda \in \theta\} \tag{4.9}$$

and $a(\theta)$ to be the orthogonal complement of a_θ in $\tilde{a}$ with respect to the Cartan–Killing form, then

$$p_\theta = m_\theta \oplus a_\theta \oplus \eta_\theta \tag{4.10}$$

is a parabolic subalgebra of $\mathcal{G}$, where

$$m_\theta = \tilde{m} \oplus \eta_+(\theta) \oplus \eta_-(\theta) \oplus a(\theta). \tag{4.11}$$

A real Cartan subalgebra $\tilde{h}$ is said to be σ invariant if

$$\tilde{h} = (\tilde{h} \cap \tilde{k}) \oplus (\tilde{h} \cap \tilde{p}). \tag{4.12}$$

A parabolic subalgebra p_θ is said to be cuspidal if there exists a σ invariant real Cartan subalgebra $\tilde{h}$ such that

$$a_\theta = \tilde{h} \cap \tilde{p}. \tag{4.13}$$

This shows that the minimal parabolic subalgebra is cuspidal.

5. Iwasawa and Langlands decompositions of $D(2, 1, \alpha)$

As an illustrative example, we consider the involutive automorphism of $D(2, 1, \alpha)$ determined by any one of the three Satake superdiagrams, say (ii), from table 2. The Cartan matrix of $D(2, 1, \alpha)$ is given by

$$C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & -\alpha \\ 0 & -\alpha & 2\alpha \end{bmatrix}. \tag{5.1}$$

From the Satake superdiagram we see that the basic root automorphisms are given by

$$\begin{aligned} \sigma(\alpha_1) &= \alpha_1 \\ -\sigma(\alpha_2) &= \alpha_1 + \alpha_2 + \alpha_3 \\ \sigma(\alpha_3) &= \alpha_3. \end{aligned} \tag{5.2}$$

The positive roots of $D(2, 1, \alpha)$ are given by

$$\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}. \tag{5.3}$$

The automorphisms of other roots are determined as

$$\begin{aligned}\sigma(\alpha_1+\alpha_2) &= -(\alpha_2+\alpha_3)\\ \sigma(\alpha_1+2\alpha_2+\alpha_3) &= -(\alpha_1+2\alpha_2+\alpha_3).\end{aligned} \tag{5.4}$$

From equations (5.2) and (5.4), we see that

$$\exp\alpha(h) = +1 \qquad \text{for } \alpha_1, \alpha_2, \alpha_3 \text{ and } \alpha_1+\alpha_2 \tag{5.5}$$

$$\exp\alpha(h) = -1 \qquad \text{for } \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3 \text{ and } \alpha_1+2\alpha_2+\alpha_3. \tag{5.6}$$

So, for $D(2,1,\alpha)$ $\tilde{k}$ is given by

$$\tilde{k} = \{\mathrm{i}h_\alpha \text{ for } \alpha = \alpha_1, \alpha_2, \alpha_3 \text{ and } (e_\alpha + e_{-\alpha}), \mathrm{i}(e_\alpha - e_{-\alpha}) \text{ for } \alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2\} \tag{5.7}$$

and $\tilde{p}$ is given by

$$\tilde{p} = \{\mathrm{i}(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) \text{ for } \alpha = \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3\}. \tag{5.8}$$

We now select a maximal Abelian subalgebra $\tilde{a}$ in the vector space $\tilde{p}$. It is clear that $\tilde{a}$ is one-dimensional and may be chosen to have a basis element

$$H_1' = \mathrm{i}(e_\alpha + e_{-\alpha}) \qquad \text{with } \alpha = \alpha_1+2\alpha_2+\alpha_3. \tag{5.9}$$

Here $R_A = \{\alpha_1+2\alpha_2+\alpha_3\}$. Note that $\tilde{m}$ is two-dimensional and its basis elements are given by

$$\begin{aligned}-\mathrm{i}H_2' &= (e_{\alpha_1} + e_{-\alpha_1})\\ -\mathrm{i}H_3' &= (e_{\alpha_3} + e_{-\alpha_3}).\end{aligned} \tag{5.10}$$

Here $R_M = \{\alpha_1, \alpha_3\}$. The required inner automorphism V that maps H' into H is then given by

$$V = \pi_\alpha v_\alpha \qquad \forall\alpha \in R_A \cup R_M$$

where $v_\alpha = \exp[ad\{\mathrm{i}a_\alpha(e_\alpha - e_{-\alpha})\}]$ and

$$a_\alpha = \frac{\pi}{\{8(\alpha,\alpha)\}^{1/2}}. \tag{5.11}$$

In our case

$$V = v_{\alpha_1+2\alpha_2+\alpha_3} v_{\alpha_1} v_{\alpha_3}. \tag{5.12}$$

Applying this to the Cartan subalgebra H' of $D(2,1,\alpha)$, we obtain

$$\begin{aligned}H_1 &= -\left\{\frac{2}{(\alpha_1+2\alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3)}\right\}^{1/2} h_{\alpha_1+2\alpha_2+\alpha_3}\\ H_2 &= -\left\{\frac{2}{(\alpha_1,\alpha_1)}\right\}^{1/2} h_{\alpha_1}\\ H_3 &= -\left\{\frac{2}{(\alpha_3,\alpha_3)}\right\}^{1/2} h_{\alpha_3}.\end{aligned} \tag{5.13}$$

With respect to this Cartan subalgebra, the set of positive roots is given by

$$\Delta^+ = \{-\alpha_1, -\alpha_2, -\alpha_3, -(\alpha_1+\alpha_2), -(\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+2\alpha_2+\alpha_3)\}. \tag{5.14}$$

The sets Δ_+^+ and Δ_-^+ can similarly be written as

$$\begin{aligned}\Delta_-^+ &= \{-\alpha_1, -\alpha_3\}\\ \Delta_+^+ &= \{-\alpha_2, -(\alpha_1+\alpha_2), -(\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+2\alpha_2+\alpha_3)\}.\end{aligned} \tag{5.15}$$

For $D(2,1,\alpha)$ the basis elements of $\tilde{\tilde{n}}$ are given by $V^{-1}e_\alpha$ where $\alpha \in \Delta_+^+$ and we see that these are given by the structures $V^{-1}e_{-\alpha_2}$, $V^{-1}e_{-(\alpha_1+\alpha_2)}$, $V^{-1}e_{-(\alpha_2+\alpha_3)}$, $V^{-1}e_{-(\alpha_1+\alpha_2+\alpha_3)}$, and $V^{-1}e_{-(\alpha_1+2\alpha_2+\alpha_3)}$. These structures can be calculated explicitly by applying the properties of inner automorphism [11]. For example, the element $V^{-1}e_{-(\alpha_1+2\alpha_2+\alpha_3)}$ is given by

$$\begin{aligned}V^{-1}e_{-(\alpha_1+2\alpha_2+\alpha_3)} &= -\frac{1}{2}(e_{\alpha_1+2\alpha_2+\alpha_3} - e_{-(\alpha_1+2\alpha_2+\alpha_3)})\\ &\quad -\frac{1}{2}\mathrm{i}\left(\frac{2}{(\alpha_1+2\alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3)}\right)^{1/2}(h_{\alpha_1}+2h_{\alpha_2}+h_{\alpha_3}).\end{aligned} \tag{5.16}$$

The elements of $\tilde{n}$ can be known by considering the elements $\tilde{\tilde{n}} \cap \mathcal{G}$ and a typical element of $\tilde{n}$ corresponding to the above structure works out to be the same as above. The required Iwasawa decomposition now reads

$$D(2,1,\alpha) = \tilde{k} \oplus \tilde{a} \oplus \tilde{n}. \tag{5.17}$$

There are $2^{|m_1|}$ classes of parabolic subalgebras, where $|m_1|$ is the dimension of $\tilde{a}$. For $D(2,1,\alpha)$ we see that $|m_1| = 1$ and there will, therefore, be two parabolic subalgebras, one being the minimal parabolic and the other the algebra itself. The minimal parabolic subalgebra is given by

$$\tilde{p}_1 = \tilde{m} \oplus \tilde{a} \oplus \tilde{n} \tag{5.18}$$

where

$$\begin{aligned}\tilde{m} &= \{(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \alpha_1, \alpha_3\}\\ \tilde{\alpha} &= \{\mathrm{i}(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \alpha_1 + 2\alpha_2 + \alpha_3\}.\end{aligned}$$

The minimal parabolic subalgebra, being the cuspidal parabolic subalgebra, leads to the following definition of a series of representation

$$\rho = \operatorname{ind} G_{p_1}(\sigma \times \tau) \tag{5.19}$$

where $\sigma \in \hat{M}$, $\tau \in \hat{N}$ and $p_1 = MAN$, is the cuspidal subgroup. Note that p_1, M, A, N, G are the groups corresponding to the algebras $\tilde{p}_1, \tilde{m}, \tilde{a}, \tilde{\tilde{n}}, \tilde{\tilde{\mathcal{G}}}$, respectively. As p_1 is minimal parabolic and $\sigma = \mathbb{1}_M$, M being Abelian, ρ is irreducible.

6. Conclusion

In conclusion, we would like to append a few remarks concerning the results obtained in this framework. By invoking the idea of Satake superdiagrams and their associated root systems we have obtained the real forms of the Lie superalgebras which are in complete conformity with those of Parker [8]. To bring out the feasibility of the mechanism more concretely, we have been able to give a general prescription for carrying out the Iwasawa and Langlands decompositions of these superalgebras with a view to generating their induced representations. This has been amply corroborated in the case of $D(2,1,\alpha)$, which is considered here as an illustrative example to demonstrate the efficacy of the method of construction of its Iwasawa and Langlands decompositions.

The results presented in this paper owe their genesis to the relative simplicity and elegance of the techniques of Satake superdiagrams corresponding to the particular Lie superalgebras considered here; the involutive automorphisms having been obtained from a modified formula. The treatment is general enough to be applicable to a variety of problems. The internal consistency and power of prediction of this approach constitute important elements offering considerable temptation to seek its applications in domains hitherto unexplored. It must, however, be emphasized that the intrinsic simplicity of the Satake superdiagram technique cannot be construed to mean that the mathematical rigour has been sacrificed. This is clearly an alternative mechanism which is no less profound than the other conventional formalisms. Its added advantage, however, lies in its versatility coupled with easy and unambiguous applicability.

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